British Mathematical Olympiad

Round 1 : Wednesday, 17th January 1996

Time allowed *Three and a half hours.*

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain far more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all five problems.*
- *Each question carries 10 marks.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by the questions 1,2,3,4,5 in order.*
- *Staple all the pages neatly together in the top* left *hand corner.*

Do not turn over until **told to do so.**

British Mathematical Olympiad

1. Consider the pair of four-digit positive integers
$$(M, N) = (3600, 2500).$$
Notice that M and N are both perfect squares, with equal digits in two places, and differing digits in the remaining two places. Moreover, when the digits differ, the digit in M is exactly one greater than the corresponding digit in N. Find all pairs of four-digit positive integers (M, N) with these properties.

2. A function f is defined over the set of all positive integers and satisfies
$$f(1) = 1996$$
and
$$f(1) + f(2) + \cdots + f(n) = n^2 f(n) \quad \text{for all } n > 1.$$
Calculate the exact value of $f(1996)$.

3. Let ABC be an acute-angled triangle, and let O be its circumcentre. The circle through A, O and B is called S. The lines CA and CB meet the circle S again at P and Q respectively. Prove that the lines CO and PQ are perpendicular.
(*Given any triangle XYZ, its* **circumcentre** *is the centre of the circle which passes through the three vertices X, Y and Z.*)

4. For any real number x, let $[x]$ denote the greatest integer which is less than or equal to x. Define
$$q(n) = \left[\frac{n}{[\sqrt{n}]} \right] \quad \text{for} \quad n = 1, 2, 3, \ldots .$$
Determine all positive integers n for which $q(n) > q(n+1)$.

5. Let a, b and c be positive real numbers.
(i) Prove that $4(a^3 + b^3) \geq (a + b)^3$.
(ii) Prove that $9(a^3 + b^3 + c^3) \geq (a + b + c)^3$.